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Free vibration analysis for tube-in-tube tall buildings

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Abstract

An approximate solution procedure is formulated for free vibration analysis of tube-in-tube tall buildings in this paper. The governing partial differential equation of motion is reduced to an ordinary differential equation with variable coefficients on the assumption that the transverse displacement is a harmonic vibration. A power-series solution which represents the mode shape function of tube-in-tube tall buildings is derived. Applying the boundary conditions yields the boundary value problem, from which the frequency equation is established and solved through a numerical process to determine the natural frequencies. Two numerical examples are performed and compared with results available in the published literature to show the accuracy of the proposed method. Care has been exercised to retain sufficient terms in power series in evaluating natural frequencies of accepted accuracy. The influences of the factors including flexural rigidity, mass per unit length and building height to the natural frequency are discussed. The method proposed herein enables one to calculate as an alternative the natural frequency of tube-in-tube tall buildings with good accuracy associated by calculators and hand, prior to use of the complicated computer programs.

1. Introduction

Free vibration analysis plays an important role in the structural design of tall buildings, especially for the first mode because the first mode shape is a dominant component in wind- and earthquake-induced vibrations of tall buildings. Therefore, it is important to investigate the calculating methods of natural frequencies and mode shapes for tall buildings. Many researchers in structural engineering have devoted to obtain accurate theoretical results for the free vibration of tall buildings in the past decades. Wang [1] obtained a formula directly from the fourth-order Sturm–Liouville differential equations for calculating the natural frequencies of tube-in-tube tall buildings. The variation principle is adopted to derive the fourth-order Sturm–Liouville differential equation and corresponding conditions at the end points. Wang [2] soon extended his work to modify the ODE solver program to calculate a numerical solution of eigenvalues for free vibration of tube-in-tube tall buildings.

An effective approach based on the classical power-series method (i.e. method of Frobenius) for solving ordinary differential equations having variables coefficients has widely been applied to solve similar complicated vibration problems. In recent years for instance, Eisenberger [3] adopted power-series solution in

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Nomenclature

a_n	the coefficient of the nth term of the power-series solution
C_i	arbitrary coefficients of general solution
$D_{4\times4}$	the (4×4) coefficient matrix of the unknowns
$\mathrm{EI}(x)$	the flexural rigidity of a tall building
EI_f	the flexural rigidity of outer tube
EI_j	the coefficient of the jth term for the power series of $\mathrm{EI}(x)$
EI_w	the flexural rigidity of inner tube
h	the story height
H	the total height of the building
i	an integer representing the number of terms in each series
j	an integer representing the number of term
K_f	the equivalent story shearing rigidity of the outer tube
m_j	the coefficient of the jth term for the power series of $m(x)$
$m(x)$	mass per unit length
n	an integer representing the number of term
N_j	the coefficient of the jth term for the power series of $N(x)$
$P(x,t)$	the applied external force
ω	the circular natural frequency
x	coordinate with base of the building
$y(x,t)$	transverse displacement of the building
$Y(x)$	the mode shape function
$Y^{(n)}(x)$	the nth derivatives of $Y(x)$
α	$\equiv \sqrt{K_f h\left(\frac{1}{\mathrm{EI}_f}+\frac{1}{\mathrm{EI}_w}\right)}$
ξ	nondimensional coordinates for x
λ	eigenvalues of the frequency equation $(=\omega^2)$

obtaining the vibration frequencies of beams on variable one- and two-parameter elastic foundations. Fung et al. [4] used power-series method to express the homogeneous solution in calculating the vibration frequencies of a rotating flexible arm carrying a moving mass.

In this paper, an approximate solution procedure is formulated for free vibration analysis of tube-in-tube tall buildings. The governing partial differential equation of motion is reduced to an ordinary differential equation with general variable coefficients on the assumption that the transverse displacement is a harmonic vibration. For which a power-series solution, which represents the mode shapes of tube-in-tube tall buildings is obtained. Applying the boundary conditions yields the eigen-value problem of finding the natural frequencies of tall buildings. To obtain a non-trivial solution of the system, the determinant of the matrix of the coefficients is set to zero for the natural frequencies. A numerical example of analyzing the free vibration and evaluating the natural frequencies for a tube-in-tube tall building are performed. Care has been exercised to retain sufficient terms in power series in evaluating accurate natural frequencies. Comparisons are also made with results available in the published literature to show accuracy of the method. In order to discuss the influences of the factors including flexural rigidity, mass per unit length and total height to the natural frequency of the buildings, calculation of the natural frequencies for 756 tube-in-tube tall buildings which having different number of storey, flexural stiffness and mass per unit length is achieved. The main objectives of this paper are (a) to develop a power-series solution procedure as an alternative method for free vibration analysis of tube-in-tube tall buildings. (b) To discuss the accuracy of the power-series solution with finite terms. (c) To discuss the influences of the factors including flexural rigidity, mass per unit length and total height to the natural frequency of the buildings for tube-in-tube tall buildings.

2. Problem formulation and solution

2.1. Power-series solution procedure

Consider a tall building with flexural rigidity $\mathrm{EI}(x)$, axial compression force $N(x)$, mass per unit length $m(x)$, and subjected to the applied lateral force $P(x,t)$ as shown in Fig. 1, the differential equation that governs the transverse displacement $y(x,t)$ can be expressed as a fourth-order partial differential

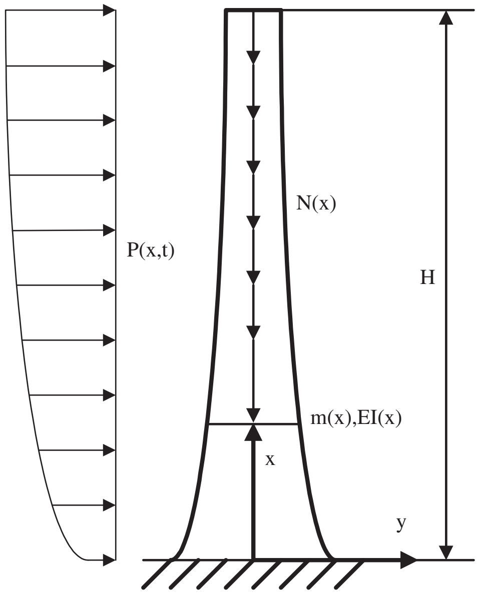

Fig. 1. A tall building model with general variable cross-section.

equation [5,9,11] as

$$\frac{\partial^2}{\partial x^2}\left(\mathrm{EI}(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right)+\frac{\partial}{\partial x}\left(N(x)\frac{\partial y(x,t)}{\partial x}\right)+m(x)\frac{\partial^2 y(x,t)}{\partial t^2}=P(x,t) \tag{1}$$

in which damping effects is neglected in Eq. (1). For analyzing free vibration of the building, one can set the applied external force $P(x,t)$ to zero. To separate the variables of Eq. (1), the transverse displacement $y(x,t)$ is assumed to be a harmonic vibration

$$y(x,t)=Y(x)\mathrm{e}^{\mathrm{i}\omega t}, \tag{2}$$

where $Y(x)$ is the mode shape function and ω is the circular natural frequency. By substituting Eq. (2) into Eq. (1), the governing equation for the mode shape function $Y(x)$ then reduces to a fourth-order ordinary differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\mathrm{EI}(x)\frac{\mathrm{d}^2 Y(x)}{\mathrm{d}x^2}\right)+\frac{\mathrm{d}}{\mathrm{d}x}\left(N(x)\frac{\mathrm{d}Y(x)}{\mathrm{d}x}\right)-\omega^2 m(x)Y(x)=0. \tag{3}$$

Its evident that Eq. (3) is a homogeneous ordinary differential equation with variable coefficients when the flexural stiffness, axial loads and mass per unit length are general functions of x. To obtain the exact solution of Eq. (3) by using ordinary trigonometric or hyperbolic functions is difficult. The problem can be solved exactly by the power-series solution method. To derive the power-series solution of Eq. (3), the coefficients in the equation are taken to have the following polynomial forms

$$\mathrm{EI}(x)=\sum_{j=0}^{i}\mathrm{EI}_j x^j,\ N(x)=\sum_{j=0}^{i}N_j x^j,\ m(x)=\sum_{j=0}^{i}m_j x^j. \tag{4}$$

Here, i is an integer representing the number of terms in each series, EI_j denotes the coefficient of the jth term for the power series of $\mathrm{EI}(x)$, N_j is the coefficient of the jth term for the power series of $N(x)$, and m_j is the coefficient of the jth term for the power series of $m(x)$. Now the solution $Y(x)$ is taken to be an infinite power series

$$Y(x)=\sum_{i=0}^{\infty}y_i x^i, \tag{5}$$

where y_i is an arbitrary coefficient. Calculating all the derivatives of Eq. (5) and substituting them into Eq. (3) yield

$$\begin{aligned}&\sum_{i=0}^{\infty}\sum_{j=0}^{i}(i-j+2)(i-j+1)(j+2)(j+1)\mathrm{EI}_{j+2}y_{i-j+2}x^i\\&\quad+\sum_{i=0}^{\infty}\sum_{j=0}^{i}2(j+1)(i-j+3)(i-j+2)(i-j+1)\mathrm{EI}_{j+1}y_{i-j+3}x^i\\&\quad+\sum_{i=0}^{\infty}\sum_{j=0}^{i}(i-j+4)(i-j+3)(i-j+2)(i-j+1)\mathrm{EI}_{j}y_{i-j+4}x^i+\sum_{i=0}^{\infty}\sum_{j=0}^{i}(j+1)(i-j+1)N_{j+1}y_{i-j+1}x^i\\&\quad+\sum_{i=0}^{\infty}\sum_{j=0}^{i}(i-j+2)(i-j+1)N_{j}y_{i-j+2}x^i-\sum_{i=0}^{\infty}\sum_{j=0}^{i}\omega^2 m_j y_{i-j}x^i=0.\end{aligned}\tag{6}$$

Using the property of identity for each term of Eq. (6), one must have

$$\begin{aligned}y_{i-j+4}=&\frac{1}{(i-j+4)(i-j+3)(i-j+2)(i-j+1)\mathrm{EI}_j}\Big(\omega^2 m_j y_{i-j}-(i-j+2)(i-j+1)(j+2)(j+1)\mathrm{EI}_{j+2}y_{i-j+2}\\&-2(j+1)(i-j+3)(i-j+2)(i-j+1)\mathrm{EI}_{j+1}y_{i-j+3}-(j+1)(i-j+1)N_{j+1}y_{i-j+1}\\&-(i-j+2)(i-j+1)N_j y_{i-j+2}\Big)\quad\text{for}\quad i=0-\infty,\quad j=0-i.\end{aligned}\tag{7}$$

Eq. (7) is the recursion relationship for every y_i. It is noted that the term of y_{i-j+4} tends to be zero as the number of terms i approaches the infinity. Thus, the general solution $y(x,\omega)$ can be obtained from Eq. (7) as

$$Y(x)=C_1Y_1(x,\omega)+C_2Y_2(x,\omega)+C_3Y_3(x,\omega)+C_4Y_4(x,\omega).\tag{8}$$

The coefficients C_i, $i = 1$–4, are arbitrary coefficients which can be obtained by applying the boundary conditions of the problem. To obtain a non-trivial solution of the system, the determinant $D_{4\times 4}$ of the matrix of the coefficients is set to zero

$$D_{4\times4}(\omega)=0.\tag{9}$$

Eq. (9) formulates in the frequency equation of free vibration for tall buildings. The natural frequencies can be obtained by solving the Eq. (9). Theoretical derivations of a tube-in-tube tall building and a cantilever one are performed in the following sections.

2.2. Free vibration analysis of tube-in-tube tall buildings

The governing equation for the transverse displacement of free vibration for a tube-in-tube tall building as shown in Fig. 2 has been derived as [5]

$$\frac{\partial^4 y(x,t)}{\partial x^4}-\alpha^2\frac{\partial^2 y(x,t)}{\partial x^2}+m\left(\frac{1}{\mathrm{EI}_w}-\frac{\alpha^2x^2}{2\mathrm{EI}}\right)\frac{\partial^2 y(x,t)}{\partial t^2}=0,\tag{10}$$

where $y(x,t)$ denotes the transverse displacement of the tube-in-tube tall building, and

$$\alpha^2=K_f h\left(\frac{1}{\mathrm{EI}_f}+\frac{1}{\mathrm{EI}_w}\right)\tag{11}$$

in which K_f is the equivalent story shearing rigidity of the outer tube, h the story height, EI_f the flexural rigidity of outer tube, EI_w the flexural rigidity of inner tube, m the mass per unit height, and EI the sum of flexural rigidity for inner tube and outer tube.

The solution of Eq. (10) for $y(x,t)$ can be expressed as a harmonic vibration

$$y(x,t)=Y(x)\mathrm{e}^{\mathrm{i}\omega t},$$

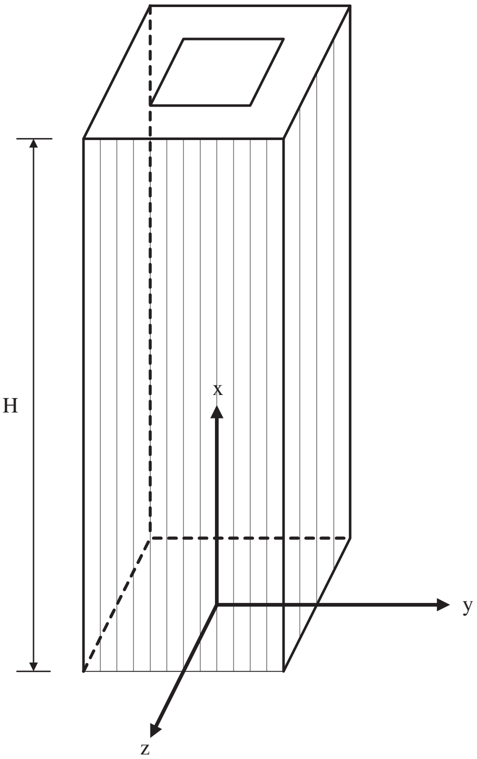

Fig. 2. A tube-in tube tall building model.

where ω is the natural frequency of the building, $Y(x)$ is the mode shape function. By substituting the expression of the solution into Eq. (10), the fourth-order ordinary differential equation governing $Y(x)$ can be extracted as

$$Y^{(4)}(x) - \alpha^2 Y''(x) - m\omega^2 \left(\frac{1}{\mathrm{EI}_w} - \frac{\alpha^2 x^2}{2\mathrm{EI}} \right) Y(x) = 0. \tag{12}$$

By introducing a non-dimensional axial coordinate in x direction

$$\xi = \frac{x}{H} (0 \leqslant \xi \leqslant 1). \tag{13}$$

And inserting Eq. (13) into Eq. (12), one can obtain

$$Y^{(4)}(\xi) - \alpha^2 H^2 Y''(\xi) - mH^4\omega^2 \left(\frac{1}{\mathrm{EI}_w} - \frac{\alpha^2 H^2 \xi^2}{2\mathrm{EI}} \right) Y(\xi) = 0, \tag{14}$$

where H denotes the total height of the building. It is obvious that Eq. (14) is a fourth-order ordinary differential equation with variable coefficients and $\xi = 0$ is an ordinary point of the equation. Hence, the power-series solution method can be applied to derive the solution of $Y(\xi)$ with the independent variable ξ. Let us assume

$$Y(\xi) = \sum_{n=0}^{\infty} a_n \xi^n \quad (0 \leqslant \xi \leqslant 1) \tag{15}$$

then

$$Y''(\xi) = \sum_{n=2}^{\infty} n(n-1) a_n \xi^{n-2}, \quad Y^{(4)}(\xi) = \sum_{n=4}^{\infty} n(n-1)(n-2)(n-3) a_n \xi^{n-4}. \tag{16a,b}$$

By substituting Eqs. (15) and (16a,b) into Eq. (14) and shifting the indices, one can obtain the series expression of the ordinary differential Eq. (14) as

$$\sum_{n=4}^{\infty} n(n-1)(n-2)(n-3)a_n\xi^{n-4} - \alpha^2H^2\sum_{n=4}^{\infty}(n-2)(n-3)a_{n-2}\xi^{n-4} - \frac{mH^4\lambda}{\mathrm{EI}_w}\sum_{n=4}^{\infty}a_{n-4}\xi^{n-4} + \frac{mH^6\alpha^2\lambda}{2\mathrm{EI}}\sum_{n=6}^{\infty}a_{n-6}\xi^{n-4} = 0, \tag{17}$$

where $\lambda = \omega^2$. Using the property of identity, for $n = 4$, the coefficient a_4 can be obtained as

$$a_4 = \frac{1}{4!}\left(\frac{mH^4\lambda}{\mathrm{EI}_w}a_0 + 2\alpha^2H^2a_2\right). \tag{18}$$

Similarly, for $n = 5$, the coefficient a_5 can be achieved as

$$a_5 = \frac{1}{5!}\left(\frac{mH^4\lambda}{\mathrm{EI}_w}a_1 + 6\alpha^2H^2a_3\right) \tag{19}$$

and the recursion relationships for any $n \geqslant 6$ can be determined as

$$a_n = \frac{1}{n(n-1)(n-2)(n-3)}\left(\frac{mH^4\lambda}{\mathrm{EI}_w}a_{n-4} + (n-2)(n-3)\alpha^2H^2a_{n-2} - \frac{mH^6\alpha^2\lambda}{2\mathrm{EI}}a_{n-6}\right). \tag{20}$$

From Eq. (20), the general solution for $Y(\xi)$ can be obtained as

$$Y(\xi) = C_1Y_1(\xi,\lambda) + C_2Y_2(\xi,\lambda) + C_3Y_3(\xi,\lambda) + C_4Y_4(\xi,\lambda), \tag{21}$$

where

$$\begin{aligned}
Y_1(\xi,\lambda) &= 1 + \frac{mH^4\lambda}{4!\mathrm{EI}_w}\xi^4 + \frac{1}{360}\left(\frac{\alpha^2H^6m\lambda}{2\mathrm{EI}_w} - \frac{\alpha^2H^6m\lambda}{2\mathrm{EI}}\right)\xi^6 + \cdots,\\
Y_2(\xi,\lambda) &= \xi + \frac{mH^4\lambda}{5!\mathrm{EI}_w}\xi^5 + \frac{1}{840}\left(\frac{\alpha^2H^6m\lambda}{6\mathrm{EI}_w} - \frac{\alpha^2H^6m\lambda}{2\mathrm{EI}}\right)\xi^7 + \cdots,\\
Y_3(\xi,\lambda) &= \xi^2 + \frac{2\alpha^2H^2}{4!}\xi^4 + \frac{1}{360}\left(\frac{mH^4\lambda}{\mathrm{EI}_w} + \alpha^4H^4\right)\xi^6 + \cdots,\\
Y_4(\xi,\lambda) &= \xi^3 + \frac{6\alpha^2H^2}{5!}\xi^5 + \frac{1}{840}\left(\frac{mH^4\lambda}{\mathrm{EI}_w} + \alpha^4H^4\right)\xi^7 + \cdots.
\end{aligned} \tag{22a-d}$$

It is noted that only the first three terms of the solution are figured out in Eq. (22). In order to discuss the rate of convergence of the power-series solution, a further study including the higher-order terms of the solution is derived in Section 3.

To determine the natural frequencies of free vibration for a tube-in-tube tall building the boundary conditions are discussed. With the structure assumed to be fixed at the base such that the cross-section of the tube does not rotate at its bottom, the boundary conditions at the base are

$$Y(0) = 0,\ Y'(0) = 0. \tag{23a-b}$$

Meanwhile, since the bending moment is only that of the inner tube at the top, the corresponding boundary condition is

$$Y''(1) = 0. \tag{24}$$

Furthermore, because the shear forces on the inner and outer tubes at any level along the structural height are in equilibrium, it yields [1]

$$Y'''(1) - \alpha^2H^2Y'(1) = 0. \tag{25}$$

From Eqs. (23) to (25), the characteristic equation of the boundary value problem can be derived for obtaining non-trivial solutions as

$$\begin{vmatrix} Y_1(0,\lambda) & Y_2(0,\lambda) & Y_3(0,\lambda) & Y_4(0,\lambda) \\ Y_1'(0,\lambda) & Y_2'(0,\lambda) & Y_3'(0,\lambda) & Y_4'(0,\lambda) \\ Y_1''(1,\lambda) & Y_2''(1,\lambda) & Y_3''(1,\lambda) & Y_4''(1,\lambda) \\ Y_1'''(1,\lambda)-\alpha^2H^2Y_1'(1,\lambda) & Y_2'''(1,\lambda)-\alpha^2H^2Y_2'(1,\lambda) & Y_3'''(1,\lambda)-\alpha^2H^2Y_3'(1,\lambda) & Y_4'''(1,\lambda)-\alpha^2H^2Y_4'(1,\lambda) \end{vmatrix} = 0. \tag{26}$$

Eq. (26) is the frequency equation for free vibration of a tube-in-tube tall building. The natural frequency can be determined by solving Eq. (26).

3. Numerical examples

3.1. Free vibration analysis of a tube-in-tube tall building

A typical tube-in-tube tall building studied by Wang [1] is adopted as a numerical example for the present study to check the accuracy of the proposed method. The plan layout of the building is shown in Fig. 3. The flexural rigidity of the outer tube is $EI_f = 35.2872\times10^9\,\mathrm{kN\,m^2}$, the flexural rigidity of the inner tube is $EI_w = 7.5538\times10^9\,\mathrm{kN\,m^2}$, mass per unit length is $m = 325.828\,\mathrm{t/m}$, the total height of the building is $H = 75.9\,\mathrm{m}$, and the value of α is $\alpha = 0.0133974546$. By inserting these data into Eq. (22), we obtain

$$\begin{aligned} Y_1(\xi,\lambda) &= 1 + 0.059645651\lambda\xi^4 + 1.693202737\times10^{-3}\lambda\xi^6 + \cdots, \\ Y_2(\xi,\lambda) &= \xi + 0.011929130\lambda\xi^5 + 1.382803942\times10^{-4}\lambda\xi^7 + \cdots, \\ Y_3(\xi,\lambda) &= \xi^2 + 0.086168175\xi^4 + 3.976376729\times10^{-3}\lambda\xi^6 + 2.969981738\times10^{-3}\xi^6 + \cdots, \\ Y_4(\xi,\lambda) &= \xi^3 + 0.051700905\xi^5 + 1.704161455\times10^{-3}\lambda\xi^7 + 1.272849316\times10^{-3}\xi^7 + \cdots. \end{aligned} \tag{27a−d}$$

Substituting Eq. (27) into Eq. (26) and solving the frequency equation through a numerical process yield $\omega_1 = 3.5180\,\mathrm{rad/s}$ and $\omega_2 = 20.7630\,\mathrm{rad/s}$. The values of natural frequencies obtained herein by the proposed method are compared with the ones found by solving the Sturm–Liouville equation [1], the mode superposition method [1] and the empirical method [6], respectively, in Table 1.

It is noted that Eq. (27a–d) figures merely out the first three terms of the power-series solution. In order to show the improvement of the proposed procedure compared to published results the rate of convergence of the power-series solution is checked. The mode shape functions including the first six terms are derived and shown

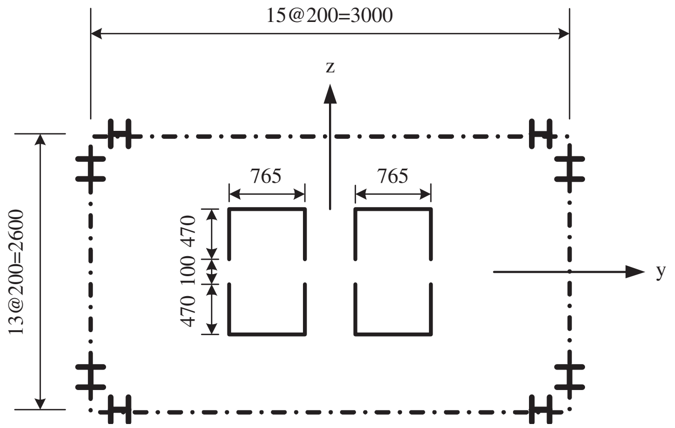

Fig. 3. Plan layout of the tube-in tube tall building studied (unit:cm).

Table 1
Natural frequencies (rad/s) of a tube-in-tube tall building with different approaches

Methods	Present	S.-L. equation	M.S.M.	Empirical
ω_1	3.5180	3.4620	3.2785	3.1566
ω_2	20.7630	21.5250	17.9212	—

Table 2
Natural frequencies obtained for different number of terms (rad/s)

Number of terms	2	3	4	5	6
ω_1	3.8806	3.5180	3.4797	3.4662	3.4641
Error (%)	12.09	1.62	0.51	0.12	0.06

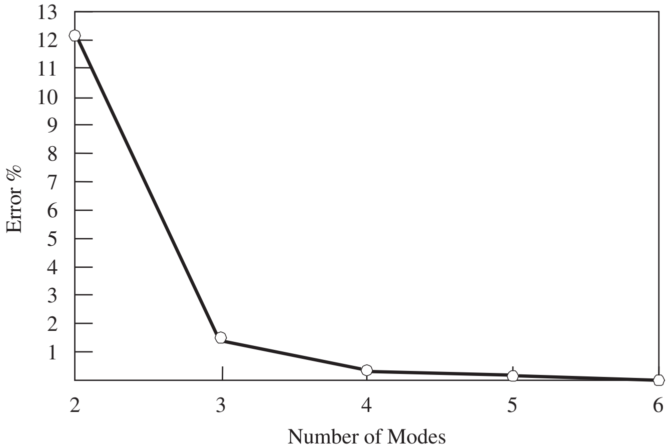

Fig. 4. Accuracy of the natural frequency.

in Appendix A. Thereby, the series are approximated by a finite number of N terms ($2 \leqslant N \leqslant 6$) and the error for the natural frequency of the first mode according to

$$\text{error}\,(N) = \left|\frac{\omega_1^N - \omega_{1,\text{ref}}}{\omega_{1,\text{ref}}}\right| \times 100 \tag{28}$$

is calculated for each number of N. In Eq. (28), the superscript $(\ )^N$ denotes the number of terms used in the approximation of the corresponding expression, and $\omega_{1,\text{ref}}$ is the reference solution.The error is shown in Table 2 and Fig. 4.

3.2. Free vibration analysis of tube-in-tube tall buildings for different flexural rigidity, mass per unit length and building height

To discuss the influences of the factors including flexural rigidity, mass per unit length and total height to the natural frequency of the buildings, a total number of 756 tube-in-tube tall buildings have been analyzed in this study. The total heights of the buildings are from 60 to 120 m. The flexural stiffness of the inner tube varies from 5×10^9 to $10 \times 10^9\,\text{kN}\,\text{m}^2$ and from 30×10^9 to $55 \times 10^9\,\text{kN}\,\text{m}^2$ for outer tube. As a result, the sum of

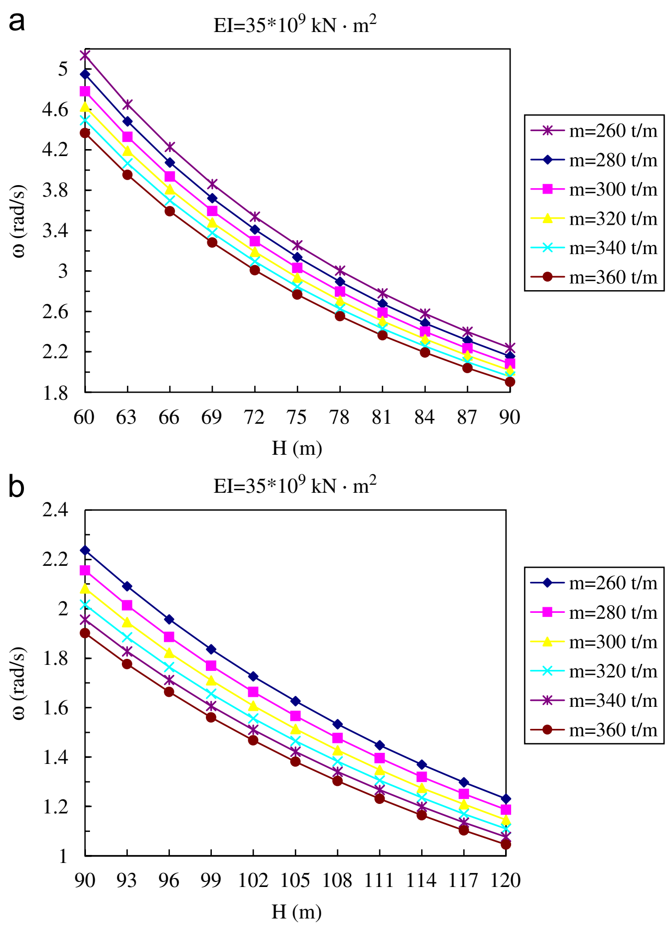

Fig. 5. Natural frequencies for $\mathrm{EI} = 35\times10^9\,\mathrm{kN\,m^2}$ ($\mathrm{EI}_w = 5\times10^9\,\mathrm{kN\,m^2}$, $\mathrm{EI}_f = 30\times10^9\,\mathrm{kN\,m^2}$).

flexural rigidity for inner tube and outer tube are between 35×10^9 and $65\times10^9\,\mathrm{kN\,m^2}$. Meanwhile, the mass per unit length is between 260 and 360 t/m. The natural frequencies obtained for different flexural stiffness are shown in Figs. 5–10. On the other hand, the relationship between the natural frequencies and flexural stiffness are shown in Figs. 11–13. Finally, the influences of mass per unit length to the natural frequencies are shown in Figs. 14–16.

3.3. Free vibration analysis of a building with generally variable cross-section

For the building with generally variable cross-section as shown in Fig. 1, a power function expression of flexural stiffness, axial loads and mass per unit length which has been discussed in literature [7] is selected as an example

$$\mathrm{EI}(x) = \mathrm{EI}_0(1+\beta x)^{m+2}, N(x) = N_0(1+\beta x)^{m+1}, m(x) = m_0(1+\beta x)^m. \tag{29}$$

By substituting Eq. (29) into Eq. (3), one can obtain

$$\mathrm{EI}_0\frac{\mathrm{d}^2}{\mathrm{d}x^2}(1+\beta x)^{m+2}\frac{\mathrm{d}^2Y(x)}{\mathrm{d}x^2} + N_0\frac{\mathrm{d}}{\mathrm{d}x}\left((1+\beta x)^{m+1}\frac{\mathrm{d}Y(x)}{\mathrm{d}x}\right) - m_0(1+\beta x)^m\omega^2Y(x) = 0. \tag{30}$$

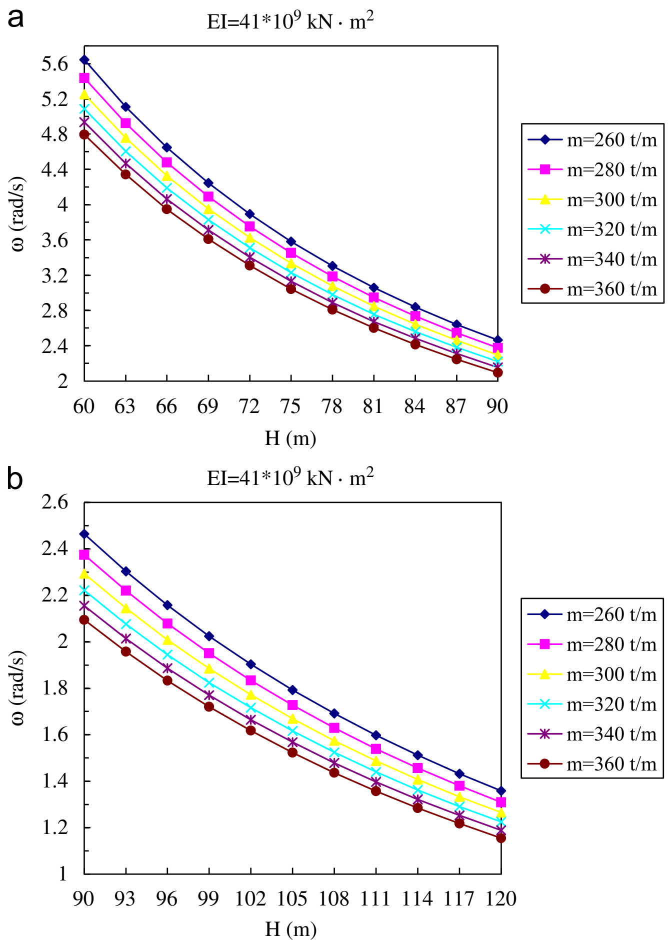

Fig. 6. Natural frequencies for $\mathrm{EI} = 41\times10^9\,\mathrm{kN\,m^2}$ ($\mathrm{EI}_w = 6\times10^9\,\mathrm{kN\,m^2}$, $\mathrm{EI}_f = 35\times10^9\,\mathrm{kN\,m^2}$).

Eq. (30) can be expressed in power series form as

$$\begin{aligned}&\mathrm{EI}_0\beta^4\sum_{n=0}^{\infty}(n+r)(n+r-1)(n+r-2)(n+r-3)a_n\eta^{n+r+m-2} + 2\mathrm{EI}_0\beta^4(m+2)\sum_{n=0}^{\infty}(n+r)(n+r-1)\\&\times(n+r-2)a_n\eta^{n+r+m-2} + \mathrm{EI}_0\beta^4(m+2)(m+1)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n\eta^{n+r+m-2} + N_0\beta^2\\&\times(m+1)\sum_{n=1}^{\infty}(n+r-1)a_{n-1}\eta^{n+r+m-2} + N_0\beta^2(m+1)\sum_{n=1}^{\infty}(n+r-1)(n+r-2)\\&\times a_{n-1}\eta^{n+r+m-2} - m_0\lambda\sum_{n=2}^{\infty}a_{n-2}\eta^{n+r+m-2} = 0. \qquad (31)\end{aligned}$$

where $\lambda=\omega^2$, $\xi = x/H$, and $\eta = (1+\beta H\xi)$. The power-series solution of mode shape functions can be obtained from solving Eq. (31) as

$$Y_1(\xi,\lambda) = \left[(1+\beta H\xi) - \frac{N_0}{4\mathrm{EI}_0\beta^2}(1+\beta H\xi)^2 + \frac{(m_0\lambda + N_0^2/\mathrm{EI}_0)}{36\mathrm{EI}_0\beta^4}(1+\beta H\xi)^3 - \frac{2m_0N_0\lambda + N_0^3/\mathrm{EI}_0}{576(\mathrm{EI}_0)^2\beta^6}(1+\beta H\xi)^4 + \cdots\right],$$

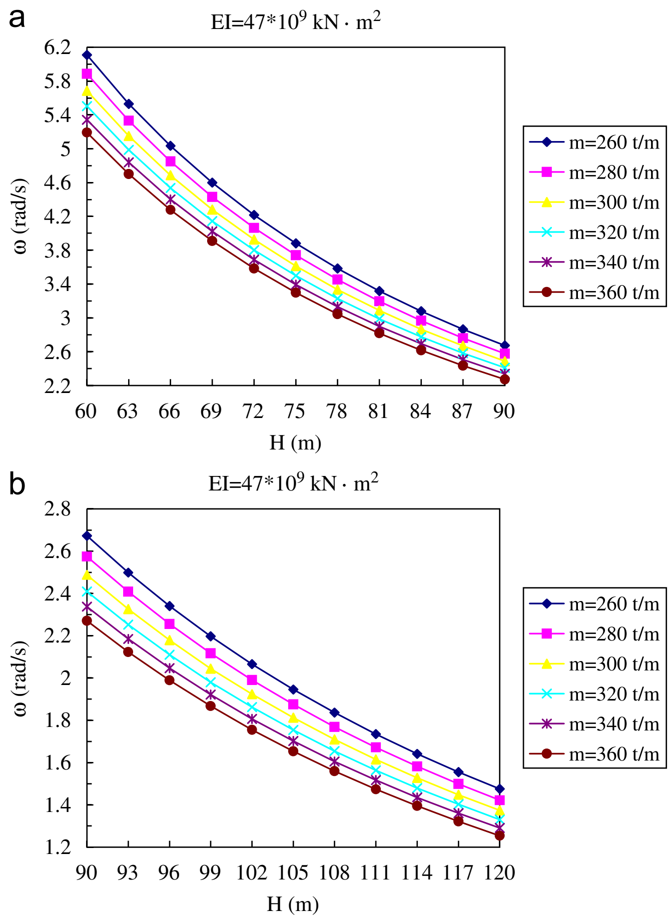

Fig. 7. Natural frequencies for $\mathrm{EI} = 47 \times 10^9\,\mathrm{kN\,m^2}$ ($\mathrm{EI}_w = 7 \times 10^9\,\mathrm{kN\,m^2}$, $\mathrm{EI}_f = 40 \times 10^9\,\mathrm{kN\,m^2}$).

$$Y_2(\xi,\lambda) = \left[1 - \frac{N_0}{\mathrm{EI}_0\beta^2}(1+\beta H\xi) + \frac{(m_0\lambda + N_0^2/\mathrm{EI}_0)}{4\mathrm{EI}_0\beta^4}(1+\beta H\xi)^2 - \frac{2m_0N_0\lambda + N_0^3/\mathrm{EI}_0}{36(\mathrm{EI}_0)^2\beta^6}(1+\beta H\xi)^3 + \cdots\right],$$

$$\begin{aligned} Y_3(\xi,\lambda) = {} & \ln(1+\beta H\xi)\left[(1+\beta H\xi) - \frac{N_0}{4\mathrm{EI}_0\beta^2}(1+\beta H\xi)^2 + \frac{(m_0\lambda + N_0^2/\mathrm{EI}_0)}{36\mathrm{EI}_0\beta^4}(1+\beta H\xi)^3 - \frac{2m_0N_0\lambda + N_0^3/\mathrm{EI}_0}{576(\mathrm{EI}_0)^2\beta^6}(1+\beta H\xi)^4 + \cdots\right] \\ & + \left[\frac{N_0}{4\mathrm{EI}_0\beta^2}(1+\beta H\xi)^2 - \frac{5(m_0\lambda + N_0^2/\mathrm{EI}_0)}{108\mathrm{EI}_0\beta^4}(1+\beta H\xi)^3 + \frac{13(2m_0N_0\lambda + N_0^3/\mathrm{EI}_0)}{3456(\mathrm{EI}_0)^2\beta^6}(1+\beta H\xi)^4 + \cdots\right], \end{aligned}$$

$$\begin{aligned} Y_4(\xi,\lambda) = {} & \ln(1+\beta H\xi)\left[1 - \frac{N_0}{\mathrm{EI}_0\beta^2}(1+\beta H\xi) + \frac{(m_0\lambda + N_0^2/\mathrm{EI}_0)}{4\mathrm{EI}_0\beta^4}(1+\beta H\xi)^2 - \frac{2m_0N_0\lambda + N_0^3/\mathrm{EI}_0}{36(\mathrm{EI}_0)^2\beta^6}(1+\beta H\xi)^3 + \cdots\right] \\ & + \left[\frac{2N_0}{\mathrm{EI}_0\beta^2}(1+\beta H\xi) - \frac{3(m_0\lambda + N_0^2/\mathrm{EI}_0)}{4\mathrm{EI}_0\beta^4}(1+\beta H\xi)^2 + \frac{11(2m_0N_0\lambda + N_0^3/\mathrm{EI}_0)}{108(\mathrm{EI}_0)^2\beta^6}(1+\beta H\xi)^3 + \cdots\right]. \end{aligned}$$

(32a–d)

Since the structures are assumed be fixed at the base such that the cross-section of the building does not rotate at its bottom, the boundary conditions at the base are [7,10]

$$Y(0) = 0, \quad Y'(0) = 0. \tag{33a–b}$$

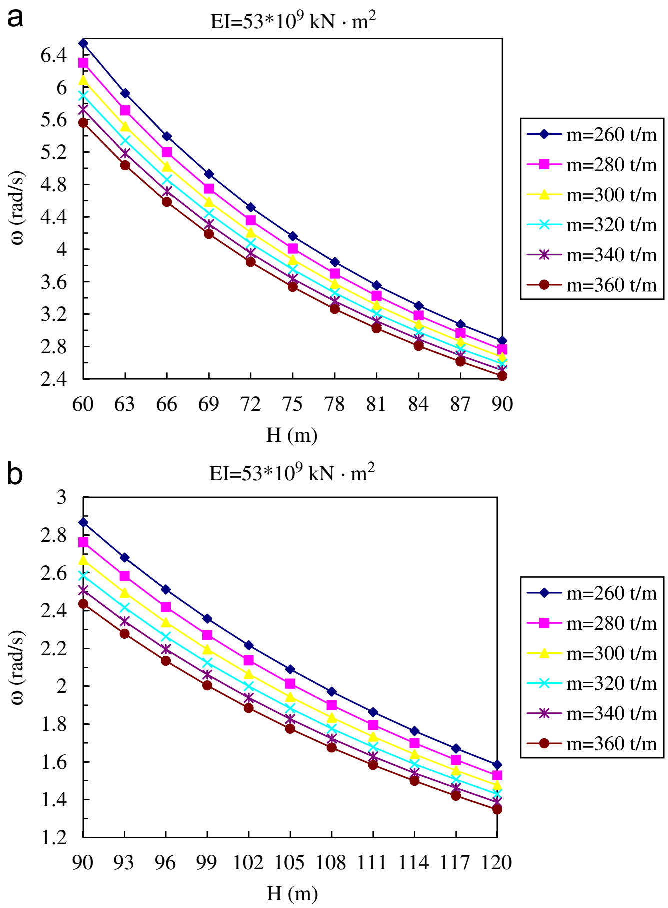

Fig. 8. Natural frequencies for $\mathrm{EI} = 53 \times 10^9\,\mathrm{kN\,m^2}$ ($\mathrm{EI}_w = 8 \times 10^9\,\mathrm{kN\,m^2}$, $\mathrm{EI}_f = 45 \times 10^9\,\mathrm{kN\,m^2}$).

On the other hand, the bending moment and the shear force is free at the top, the corresponding boundary conditions are

$$\mathrm{EI}\,Y''(1) + NY(1) = 0, \quad (\mathrm{EI}\,Y''(1) + NY(1))' = 0. \tag{34a−b}$$

The frequency equation of the building can be expressed as

$$\begin{vmatrix} Y_1(0,\lambda) & Y_2(0,\lambda) & Y_3(0,\lambda) & Y_4(0,\lambda) \\ Y_1'(0,\lambda) & Y_2'(0,\lambda) & Y_3'(0,\lambda) & Y_2'(0,\lambda) \\ \mathrm{EI}\,Y_1''(1,\lambda) + NY_1(1,\lambda) & \mathrm{EI}\,Y_2''(1,\lambda) + NY_2(1,\lambda) & \mathrm{EI}\,Y_3''(1,\lambda) + NY_3(1,\lambda) & \mathrm{EI}\,Y_4''(1,\lambda) + NY_4(1,\lambda) \\ \left(\mathrm{EI}\,Y_1''(1,\lambda) + NY_1(1,\lambda)\right)' & \left(\mathrm{EI}\,Y_2''(1,\lambda) + NY_2(1,\lambda)\right)' & \left(\mathrm{EI}\,Y_3''(1,\lambda) + NY_3(1,\lambda)\right)' & \left(\mathrm{EI}\,Y_4''(1,\lambda) + NY_4(1,\lambda)\right)' \end{vmatrix} = 0 \tag{35}$$

A typical shear-wall tall building with 27-storeys studied by Li [7] is adopted as a numerical example. Based on the full-scale measurement of free vibration [8], this building can be treated as a cantilever bar with variable cross-section under various axial loads. Because the variation is comparatively small, the mass per unit length is reasonably assumed uniformly distributed along the height of the building. One can obtain

$$m(x) = m_0(1 + \beta x)^0. \tag{36}$$

The distributions of flexural stiffness and axial loading are assumed as

$$N(x) = N_0(1 + \beta x)^1, \quad \mathrm{EI}(x) = \mathrm{EI}_0(1 + \beta x)^2, \tag{37a−b}$$

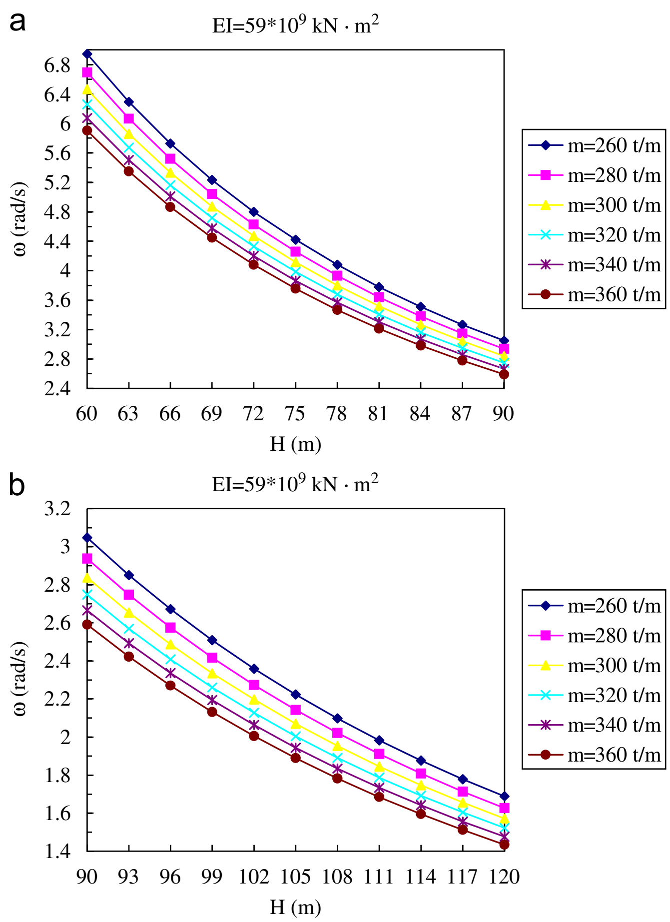

Fig. 9. Natural frequencies for $\mathrm{EI} = 59 \times 10^9\,\mathrm{kN\,m^2}$ ($\mathrm{EI}_w = 9 \times 10^9\,\mathrm{kN\,m^2}$, $\mathrm{EI}_f = 50 \times 10^9\,\mathrm{kN\,m^2}$).

where $m_0 = 38014.2\,\mathrm{kg/m}$, $N_0 = 9301390.7\,\mathrm{N}$, $\mathrm{EI}_0 = 60.38 \times 10^{13}\,\mathrm{N\,m^2}$, $\beta = -3.796 \times 10^{-3}$, $H = 76.0\,\mathrm{m}$. By inserting these data into Eq. (32) and Eq. (35) and solving Eq. (35) through a numerical process, we obtain $\omega_1 = 6.5822\,\mathrm{rad/s}$. The value of natural frequency is compared with the ones found by Li [7] and the measured value [8] in Table 3.

4. Results and discussions

4.1. Accuracy of the proposed method

It can be seen in Table 1 that the results calculated by the proposed method agree very well with the results found by other published methods. The differences of the natural frequencies of the tube-in-tube tall building between the proposed method and other published methods are small. In fact, the difference between the natural frequency of the first mode calculated by the proposed method and from solving the Sturm–Liouville equation is about 1.62% and 3.54% for the second mode. On the other hand, the difference of the natural frequencies between the proposed method and other published methods is about 1.02% for the cantilever model of tall building studied. Although larger differences are found between the results of the mode superposition method and the proposed method, the agreement is still satisfactory sufficiently for structural design.

For the tube-in-tube tall building studied in this paper, it can be seen from Fig. 4 that the error of the approximate solution is about 12.09% when the first two terms of the power-series solution are

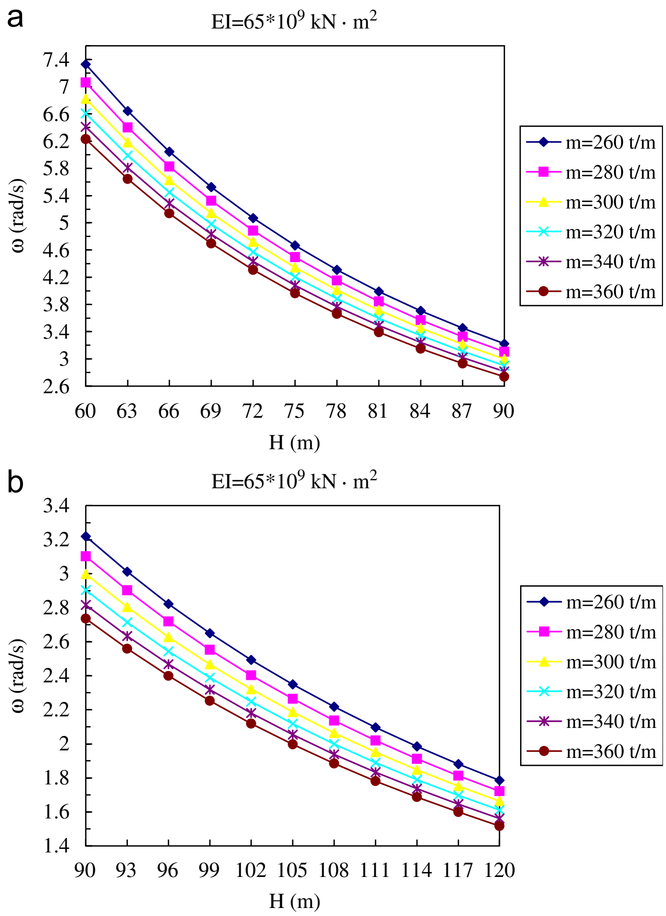

Fig. 10. Natural frequencies for $\mathrm{EI} = 65 \times 10^9\,\mathrm{kN\,m^2}$ ($\mathrm{EI}_w = 10 \times 10^9\,\mathrm{kN\,m^2}$, $\mathrm{EI}_f = 55 \times 10^9\,\mathrm{kN\,m^2}$).

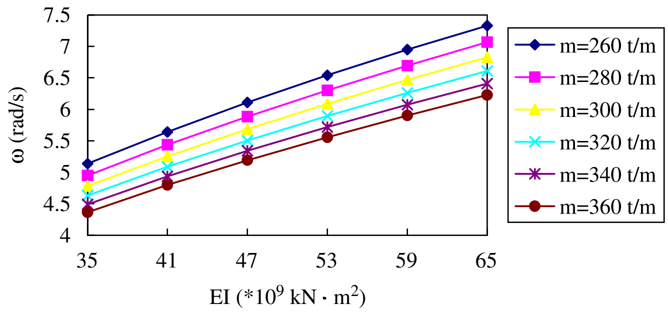

Fig. 11. Comparison of natural frequencies for different EI ($H = 60\,\mathrm{m}$).

merely considered. Meanwhile, if we consider the first three terms of the power-series solution, the error of the approximate solution reduces significantly to about 1.62%. Furthermore, the errors are about 0.51%, 0.12% and 0.06% as the first four terms, five terms and six terms are considered, respectively. Therefore, its of good accuracy to calculate the natural frequencies by using the first three terms of the power-series solution.

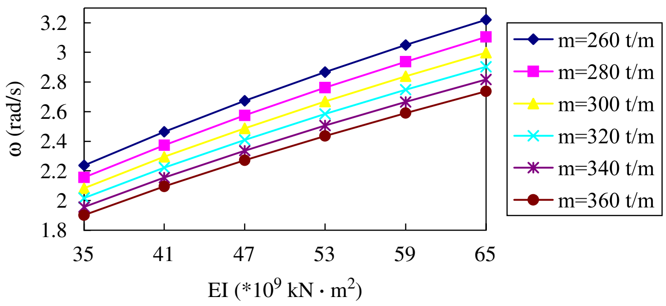

Fig. 12. Comparison of natural frequencies for different EI ($H = 90$ m).

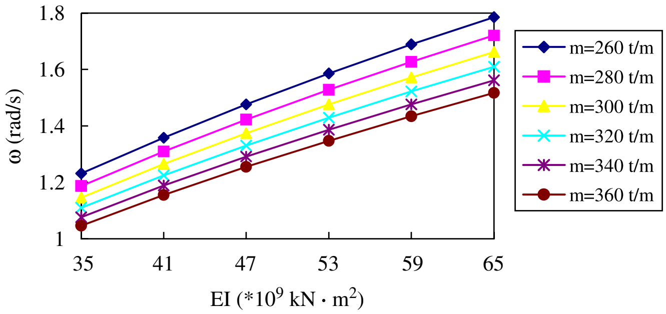

Fig. 13. Comparison of natural frequencies for different EI ($H = 120$ m).

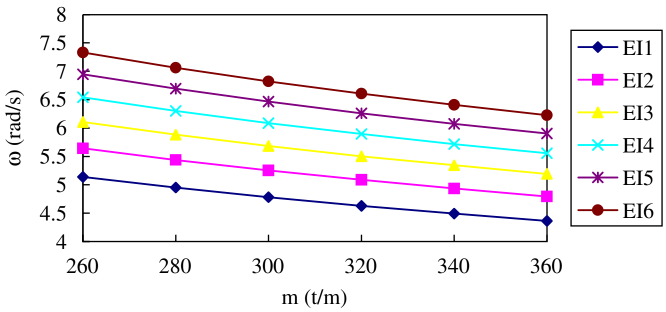

Fig. 14. Comparison of natural frequencies for different mass per unit length ($H = 60$ m).

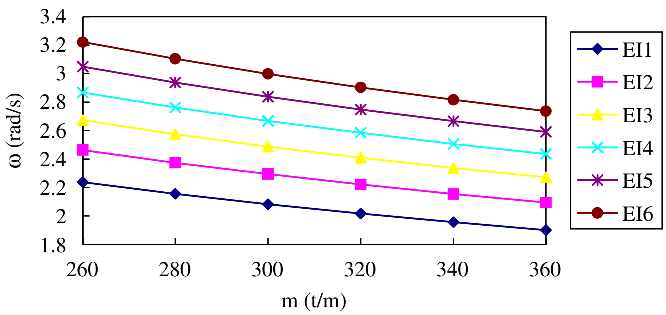

Fig. 15. Comparison of natural frequencies for different mass per unit length ($H = 90$ m).

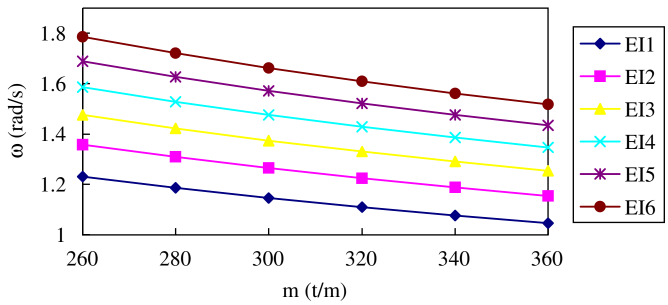

Fig. 16. Comparison of natural frequencies for different mass per unit length ($H = 120$ m).

Table 3
Natural frequencies obtained for the building studied by different methods (rad/s)

Methods	Present	Q.S. Li	Measured
ω_1	6.5822	6.5170	6.4775

4.2. Discussion on the results

As can be seen from Figs. 5 to 10 the value of the natural frequencies decreases as the total height of the buildings increase. However, it is observed from Figs. 11 to 13 that the value of the natural frequencies increases as the flexural stiffness of the buildings increases. Meanwhile, as the value of mass per unit length increases the natural frequencies decreases as can be seen from Figs. 14 to 16.

Figs. 5–10 are the natural frequencies of tube-in-tube tall buildings with total height from 60 to 120 m. One can calculate approximately the natural frequency of an individual tube-in-tube tall building by looking up these diagrams. For example, a tube-in-tube tall building with total height $H = 110$ m, flexural stiffness of the inner tube $EI_w = 8 \times 10^9$ kN m^2, flexural stiffness of the outer tube $EI_f = 45 \times 10^9$ kN m^2, and mass per unit length $m = 330$ t/m, the natural frequency can be obtained from Fig. 8 with interpolation. In fact, the value of the natural frequency obtained from Fig. 8 is 1.6847 rad/s.

5. Conclusions

The formulae proposed in this paper can be used as an alternative to determine the natural frequencies of tube-in-tube tall buildings. Two numerical examples have been performed and compared to the published results to demonstrate the accuracy of the method. Calculation of the natural frequencies for 756 tube-in-tube tall buildings which having different number of storey, flexural stiffness and mass per unit length is achieved. The following concluding remarks could be drawn from the present study:

(1) An approximate solution procedure is formulated in this paper for free vibration analysis of tube-in-tube tall buildings.
(2) The method proposed herein enables one to calculate the natural frequency rapidly with accepted accuracy associated by calculators and hand, prior to the use of the complicated computer programs.
(3) The influences of the factors including flexural rigidity, mass per unit length and total height of the building to the natural frequency of the tube-in-tube tall buildings are discussed.
(4) It's of good accuracy to calculate the natural frequencies by using the first three terms of the power-series solution. The natural frequencies found by the proposed method agree very well with the results found by other published methods.

(5) The method proposed herein can be adopted as an alternative procedure to evaluate the natural frequencies for free vibration of tube-in-tube buildings in the preliminary stage of structural design.

Acknowledgment

The financial support from the National Science Council of the Republic of China (NSC 94-2625-Z-390-002-) for conducting the present investigation is gratefully acknowledged.

Appendix A

$$\begin{aligned} Y_1(\xi,\lambda) = {} & 1 + \frac{mH^4\lambda}{4!\mathrm{EI}_w}\xi^4 + \frac{1}{360}\left(\frac{\alpha^2H^6m\lambda}{2\mathrm{EI}_w} - \frac{\alpha^2H^6m\lambda}{2\mathrm{EI}}\right)\xi^6 \\ & + \frac{1}{40320}\left(\frac{m^2H^8\lambda^2}{\mathrm{E}^2\mathrm{I}_w^2} + \frac{m^2H^8\alpha^4\lambda}{\mathrm{EI}_w} - \frac{mH^8\alpha^4\lambda}{\mathrm{EI}}\right)\xi^8 \\ & + \left(\frac{m^2H^{10}\alpha^2\lambda^2}{1814400\mathrm{E}^2\mathrm{I}_w^2} + \frac{mH^{10}\alpha^6\lambda}{3628800\mathrm{EI}_w} - \frac{mH^{10}\alpha^6\lambda}{3628800\mathrm{EI}} - \frac{m^2H^{10}\alpha^2\lambda^2}{226800\mathrm{E}^2\mathrm{II}_w}\right)\xi^{10} \\ & + \left(\frac{m^3H^{12}\lambda^3}{479001600\mathrm{E}^3\mathrm{I}_w^3} + \frac{m^2H^{12}\alpha^4\lambda^2}{159667200\mathrm{E}^2\mathrm{I}_w^2} - \frac{m^2H^{12}\alpha^4\lambda^2}{10644480\mathrm{E}^2\mathrm{II}_w}\right. \\ & \left. + \frac{m^2H^{12}\alpha^4\lambda^2}{1710200\mathrm{E}^2\mathrm{I}^2} + \frac{mH^{12}\alpha^8\lambda}{479001600\mathrm{EI}_w} - \frac{mH^{12}\alpha^8\lambda}{479001600\mathrm{EI}}\right)\xi^{12} + \cdots, \end{aligned} \tag{A.1}$$

$$\begin{aligned} Y_2(\xi,\lambda) = {} & \xi + \frac{mH^4\lambda}{5!\mathrm{EI}_w}\xi^5 + \frac{1}{840}\left(\frac{\alpha^2H^6m\lambda}{6\mathrm{EI}_w} - \frac{\alpha^2H^6m\lambda}{2\mathrm{EI}}\right)\xi^7 \\ & + \left(\frac{m^2H^8\lambda^2}{362880\mathrm{E}^2\mathrm{I}_w^2} + \frac{mH^8\alpha^4\lambda}{1663200\mathrm{E}^2\mathrm{II}_w} - \frac{mH^8\alpha^4\lambda}{120960\mathrm{EI}}\right)\xi^9 \\ & + \left(\frac{m^2H^{10}\alpha^2\lambda^2}{19958400\mathrm{E}^2\mathrm{I}_w^2} + \frac{mH^{10}\alpha^6\lambda}{39916800\mathrm{EI}_w} - \frac{mH^{10}\alpha^6\lambda}{13305600\mathrm{EI}} - \frac{m^2H^{10}\alpha^2\lambda^2}{1663200\mathrm{E}^2\mathrm{II}_w}\right)\xi^9 \\ & + \left(\frac{m^3H^{12}\lambda^3}{6227020800\mathrm{E}^3\mathrm{I}_w^3} + \frac{m^2H^{12}\alpha^4\lambda^2}{2075673600\mathrm{E}^2\mathrm{I}_w^2} - \frac{m^2H^{12}\alpha^4\lambda^2}{98841600\mathrm{E}^2\mathrm{II}_w}\right. \\ & \left. + \frac{m^2H^{12}\alpha^4\lambda^2}{57657600\mathrm{E}^2\mathrm{I}^2} + \frac{mH^{12}\alpha^8\lambda}{6227020800\mathrm{EI}_w} - \frac{mH^{12}\alpha^8\lambda}{2075673600\mathrm{EI}}\right)\xi^{13} + \cdots, \end{aligned} \tag{A.2}$$

$$\begin{aligned} Y_3(\xi,\lambda) = {} & \xi^2 + \frac{2\alpha^2H^2}{4!}\xi^4 + \frac{1}{360}\left(\frac{mH^4\lambda}{\mathrm{EI}_w} + \alpha^4H^4\right)\xi^6 + \left(\frac{mH^6\alpha^2\lambda}{10080\mathrm{EI}_w} + \frac{\alpha^6H^6}{20160} - \frac{mH^6\alpha^2\lambda}{3360\mathrm{EI}}\right)\xi^8 \\ & + \left(\frac{m^2H^8\lambda^2}{1814400\mathrm{E}^2\mathrm{I}_w^2} + \frac{mH^8\alpha^4\lambda}{604800\mathrm{EI}_w} - \frac{mH^8\alpha^4\lambda}{86400\mathrm{EI}} + \frac{\alpha^2H^8}{1814400}\right)\xi^{10} \\ & + \left(\frac{m^2H^{10}\alpha^2\lambda^2}{79833600\mathrm{E}^2\mathrm{I}_w^2} - \frac{17m^2H^{10}\alpha^2\lambda^2}{119750400\mathrm{E}^2\mathrm{II}_w} + \frac{mH^{10}\alpha^6\lambda}{59875200\mathrm{EI}_w} - \frac{7mH^{10}\alpha^6\lambda}{34214400\mathrm{EI}} + \frac{\alpha^{10}H^{10}}{239500800}\right)\xi^{12} + \cdots, \end{aligned} \tag{A.3}$$

$$
\begin{aligned}
Y_4(\xi,\lambda) = {} & \xi^3 + \frac{6\alpha^2H^2}{5!}\xi^5 + \frac{1}{840}\left(\frac{mH^4\lambda}{\mathrm{EI}_w} + \alpha^4H^4\right)\xi^7 + \left(\frac{mH^6\alpha^2\lambda}{30240\mathrm{EI}_w} + \frac{\alpha^6H^6}{60480} - \frac{mH^6\alpha^2\lambda}{6048\mathrm{EI}}\right)\xi^9 \\
& + \left(\frac{m^2H^8\lambda^2}{6652800\mathrm{E}^2\mathrm{I}_w^2} + \frac{mH^8\alpha^4\lambda}{2217600\mathrm{EI}_w} - \frac{31mH^8\alpha^4\lambda}{6652800\mathrm{EI}} + \frac{\alpha^2H^8}{6652800}\right)\xi^{11} \\
& + \left(\frac{m^2H^{10}\alpha^2\lambda^2}{345945600\mathrm{E}^2\mathrm{I}_w^2} - \frac{23m^2H^{10}\alpha^2\lambda^2}{518918400\mathrm{E}^2\mathrm{II}_w} + \frac{mH^{10}\alpha^6\lambda}{259459200\mathrm{EI}_w} - \frac{67mH^{10}\alpha^6\lambda}{1037836800\mathrm{EI}} + \frac{\alpha^{10}H^{10}}{1037836800}\right)\xi^{13} + \cdots .
\end{aligned}
\tag{A.4}
$$

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